18. KARAPETYAN A.V. and RUMYANTSEV V.V., Stability of conservative and dissipative systems. Science and Engineering Surveys, General Mechanics, 6, VINITI, Moscow, 1983.
19. MOSHCHUK N.K., On the motion of a heavy solid on a horizontal plane with viscous friction, PMM, 49, 1, 1985.

Translated by M.D.F.

PMM U.S.S.R., Vol. 50, No. 1, pp. 35-39, 1986
Printed in Great Britain

0021-8928/86 $10.00+0.00
© 1987 Pergamon Journals Ltd.

STABILITY UNDER CONSTANTLY ACTING PERTURBATIONS, AND AVERAGING IN AN UNBOUNDED INTERVAL IN SYSTEMS WITH IMPULSES*

V.SH. BURD

The question of the closeness of non-stationary solutions of the exact and averaged equations in an unlimited time interval is investigated for ordinary differential equations whose right sides contain generalized functions of time (generalized derivatives of functions of bounded variation). The appropriate assertions in the development of the method proposed in /1/ are derived from a special theorem on stability under permanently acting perturbations. The results obtained (more general in the case of equations with smooth coefficients then the assertions in /2, 3/) afford an apportunity for giving a foundation to the applicability of the averaging method to quasiconservative vibration impact systems /4/.

We note that the question of the correspondence between solutions of the exact equations and the stationary solutions of the average equations was investigated in /5/ (see /6/ also) for systems in standard form with impulsive action.

1. We shall use the following notation: R^n is a Euclidean n-space, $|x|$ is the norm of the element $x \in R^n$, I is the interval $[0, \infty)$, $B_x(K) = \{x: x \in R^n, |x| \leqslant K\}$, $G = I \times B_x(K)$. We shall henceforth consider integrals of the form

$$\int_{t_1}^{t_2} f(s, x(s))\, du(s), \quad (t_1, t_2) \in J \tag{1.1}$$

which are understood to be Lebesgue-Stieltjes integrals. We shall say with respect to the integrating function $u(t)$ that $u(t) \in BU(J)$ if $u(t)$ is a scalar function defined for $t \in J$ and possessing the following properties:

1) $u(t)$ is continuous on the right and is of limited variation in each compact subinterval of the interval J;

2) The discontinuities $t_1 < t_2 < \ldots$ $(t_1 \geqslant t_0 \geqslant 0)$ of the function $u(t)$ have the single limit point $+\infty$.

Functions defined on J with values in $B_x(K)$ continuous to the right and with the same points of discontinuity of the first kind as $u(t)$ will be considered as $x(t)$. Then if $f(t, x)$ is a function defined in G with values in R^n bounded in the norm, continuous in x uniformly with respect to t and having not more than a denumerable number of points of discontinuity of the first kind in t, the integral (1.1) exists. We note that with the above assumptions, the appropriate generalization of the Riemann-Stieltjes integral can be used in place of the Lebesgue-Stieltjes integral. Later, if the question of the existence of the integral (1.1) is not especially stipulated, we shall assume that the listed conditions are satisfied.

For the function $f(t, x)$ defined in G and integrable with respect to $u(t) \in BU(J)$, we introduce

$$S_x(f) = \sup_{|t_2 - t_1| \leqslant 1} \left| \int_{t_1}^{t_2} f(s, x)\, du(s) \right|, \quad x \in B_x(K)$$

Lemma 1. Let the function $f(t, x)$ be defined on G and continuous in x uniformly with respect to $t \in J$. Let the function $x(t)$ which is continuous to the right with values in

*Prikl. Matem. Mekhan., 50, 1, 50-56, 1986

$B_x(K)$ be a function of bounded variation in each compact subinterval of J and with discontinuities at the same points as the function $u(t) \in B_x(K)$. Let $f(t, x(t))$ be integrable with respect to $u(t)$. Then for any $\eta > 0$ there is an $\varepsilon > 0$ such that

$$\sup_{|t_2-t_1|\leqslant 1} \Big|\int_{t_1}^{t_2} f(s, x(s))\, du(s)\Big| < \eta, \quad (t_1, t_2) \in [0, T], \quad 0 < T < \infty$$

if $S_x(f) < \varepsilon$.

Proof. By virtue of the conditions of the lemma for each $\eta > 0$ there can be a $\delta > 0$ such that $|f(t, x_1) - f(t, x_2)| < \eta/2$ for $|x_1 - x_2| < \delta$. We denote the piecewise-constant function with values in $B_x(K)$ for which $|x(t) - x^0(t)| < \delta$, $t \in [0, T]$ by $x^0(t)$, where in each interval whose length does not exceed one, the function $x(t)$ takes on not more than k different values, where the number k depends only on δ. Let x_j $(j = 1, \ldots, k)$ be values of $x^0(t)$ in the interval $|t_2 - t_1| \leqslant 1$. We set $\varepsilon = \eta/(2k)$. Then

$$\Big|\int_{t_1}^{t_2} f(s, x(s))\, du(s)\Big| \leqslant \Big|\int_{t_1}^{t_2} [f(s, x(s)) - f(s, x^0(s))]\, du(s)\Big| +$$

$$\Big|\int_{t_1}^{t_2} f(s, x^0(s))\, du(s)\Big| \leqslant \frac{n}{2} + \sum_{j=1}^{k}\Big|\int_{t_1}^{t_2} f(s, x_j)\, du(s)\Big| \leqslant \frac{\eta}{2} + \frac{\eta}{2k} k = \eta$$

The last inequality holds for any t_1, t_2 satisfying the inequality $|t_2 - t_1| \leqslant 1$, which indeed proves the lemma.

2. We consider a differential equation in generalized functions in R^n

$$Dx(t) = X(t, x) + R(t, x)\, Du(t) \tag{2.1}$$

where the function $X(t, x)$ and $R(t, x)$ are defined in G, $u(t) \in BU(J)$, $Dx(t)$ and $Du(t)$ are generalized derivatives of the functions $x(t)$ and $u(t)$ respectively. We understand the function $x(t, t_0, x_0)$, that is continuous to the right, is of bounded variation in I and such that its generalized derivative in (t_0, T), $T \in I$ satisfies (2.1), to be a solution of (2.1) defined in the interval I with left end t_0 satisfying the condition $x(t_0) = x_0$. We know /7/ that the function $x(t)$ is a solution of (2.1) in the interval I passing through (t_0, x_0) if and only if it satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} X(s, x(s))\, ds + \int_{t_0}^{t} R(s, x(s))\, du(s), \; t \in I \tag{2.2}$$

where for each function of bounded variation $x(t)$ in I that is continuous to the right, the function $X(t, x(t))$ is integrable while $R(t, x(t))$ is integrable in I with respect to $u(t)$, where the second integral is considered in the interval $(t_0, t]$. The function $x(t)$ as a solution of (2.2) evidently has discontinuities at the same points as does $u(t)$.

In addition to (2.1), we consider an unperturbed ordinary differential equation in R^n

$$dy/dt = X(t, y) \tag{2.3}$$

We assume that (2.3) has the solution $\psi(t, t_0, \xi_0)$ $(\psi(t_0, t_0, \xi_0) = \xi_0)$, defined for all $t \geqslant t_0 \geqslant 0$, which is contained, together with its certain ρ-neighbourhood $(\rho > 0)$, in the set G.

Theorem 1. Let the function $X(t, x)$ satisfy the Lipschitz condition in x for $x \in B_x(K)$, $t \in J$ with constant L, let the function $R(t, x)$ be uniformly continuous in x with respect to $t \in J$, and let the solution $\psi(t, t_0, \xi_0)$ of (2.3) be uniformly asymptotically stable. Then for any $\varepsilon > 0$ $(0 < \varepsilon < \rho)$ numbers $\eta_1(\varepsilon)$, $\eta_2(\varepsilon)$ exist for all solutions $x(t, t_0, x_0)$ $(x(t_0, t_0, x_0) = x_0)$ of (2.1) defined for $t \geqslant t_0$ with values in $B_x(K)$ and with initial data satisfying the inequality $|x_0 - \xi_0| < \eta_1(\varepsilon)$ and for all $R(t, x)$, satisfying the inequality

$$\sup_{|t_2-t_1|\leqslant 1} \Big|\int_{t_1}^{t_2} R(t, x)\, du(t)\Big| < \eta_2(\varepsilon), \; t_1, t_2 \in J, \quad x \in B_x(K)$$

and the inequality

$$|x(t, t_0, x_0 - \psi(t, t_0, \xi_0)| < \varepsilon \tag{2.4}$$

holds for all $t \geqslant t_0$

Proof. We use the reasoning of Lemma 6.3 in Chapter III of /8/. Let $y(t, t_0, x_0)$ be the solution of (2.3) with the same initial condition as the solution $x(t, t_0, x_0)$ of (2.1). From the conditions of the thoerem we obtain the inequality

$$\Delta(t) = L\int_{t_0}^{t_1} \Delta(s)\, ds + F(t), \quad F(t) = \Big|\int_{t_0}^{t} R(s, x(s, t_0, x_0))\, du(s)\Big|$$

$$(\Delta(t) = |x(t, t_0, x_0) - y(t, t_0, x_0)|)$$

A well-known integral inequality (see /8/, say) yields

$$\Delta(t) \leqslant F(t) + L\int_{t_0}^{t} \exp(L(t-s))F(s)\,ds$$

Hence we obtain for $t_0 \leqslant t \leqslant t_0 + T$

$$\Delta(t) \leqslant (T+1)(1+LT\exp(LT)) \sup_{|t_2-t_1|\leqslant 1} \left| \int_{t_1}^{t_2} R(s, x(s, t_0, x_0))\,du(s) \right|$$

By virtue of the uniform asymptotic stability of the solution $\psi(t, t_0, \xi_0)$ of (2.3), numbers $\delta < \varepsilon$ and $T > 0$ exist such that it follows from the inequality $|x_0 - \xi_0| < \delta$ that

$$|y(t, t_0, x_0) - \psi(t, t_0, \xi_0)| < \varepsilon/2, \; t \geqslant t_0;$$
$$|y(t_0 + T, t_0, x_0) - \psi(t_0 + T, t_0, \xi_0)| < \delta/2 \tag{2.5}$$

It follows from Lemma 1 that the number $\eta_2(\varepsilon)$ can be selected in such a manner that the following inequlity is satisfied:

$$\Delta(t) < \delta/2, \; t_0 \leqslant t \leqslant t_0 + T \tag{2.6}$$

Then

$$|x(t, t_0, x_0) - \psi(t, t_0, \xi_0)| < \varepsilon/2 + \delta/2 < \varepsilon, \; t_0 \leqslant t \leqslant t_0 + T$$

Furthermore, we obtain from (2.5) and (2.6)

$$|x(t_0 + T, t_0, x_0) - \psi(t_0 + T, t_0, \xi_0)| < \delta$$

The concluding part of the proof of Theorem 1 agrees completely with the concluding part of the proof of the above-mentioned Lemma 6.3.

Remarks. 1°. It was assumed in the formulation of the thoerem that the solution $x(t, t_0, x_0)$ is defined for $t \geqslant t_0$ and lies in $B_x(K)$. If the conditions of the local existence theorem for solutions of (2.1) are satisfied and $S_x(R)$ is sufficiently small for $x \in B_x(K)$, then the solution $x(t, t_0, x_0)$ with initial condition sufficiently close, in the norm, to the initial condition of the solution $\psi(t, t_0, \xi_0)$ of (2.3), will be defined for all $t \geqslant t_0$ and will not emerge from the sphere $B_x(K)$.

2°. If the solution $\psi(t, t_0, \xi_0)$ of (2.3) is uniformly asymptotically stable in part of the variables $\psi_1, \ldots, \psi_k$ $(k > n)$, then the inequality (2.4) in the assertion of Theorem 1 is replaced by the inequality

$$|x_i(t, t_0, x_0) - \psi_i(t, t_0, \xi_0)| < \varepsilon, \; i = 1, \ldots, k$$

3. We apply Theorem 1 to the problem of taking the average in an unbounded interval. We use the following scheme. The equation to be investigated in R^n will be written in the form

$$Dx(t) = R(t, x, \varepsilon)\,Du(t, \varepsilon) \tag{3.1}$$

where ε is a small positive parameter, and it is shown that the limit equation

$$\lim_{\varepsilon\to 0} \sup_{|t_2-t_1|\leqslant 1} \left| \int_{t_1}^{t_2} R(s, x, \varepsilon)\,du(s, \varepsilon) - \int_{t_1}^{t_2} X(s, x)\,ds \right| = 0, \quad x \in B_x(K) \tag{3.2}$$

is valid, where $X(t, x)$ is the right side of the averaged ordinary differential equation in R^n. This enables us to obtain corresponding assertions about taking the average in an unbounded interval as a corollary of Theorem 1. For convenience we shall say that the right side of (3.1) converges integrally to $X(t, x)$ as $\varepsilon \to 0$ if the limit Eq.(3.2) holds.

We turn first to the differential equation in R^n by fast and slow time in the standard form

$$Dx(t) = \varepsilon X(t, \tau, x, \varepsilon)\,Du(t), \; \tau = \varepsilon t \tag{3.3}$$

where ε is a small positive parameter that varies in the interval $[0, \varepsilon_0]$, the function $X(t, \tau, x, \varepsilon)$ with values in R^n is defined for $t \in J$, $x \in B_x(K)$, $\varepsilon \in [0, \varepsilon_0]$, $u(t) \in BU(J)$ and, in addition, $u(t)$ is bounded in J.

Theorem 2. Suppose

1) the function $X(t, \tau, x, \varepsilon)$ is uniformly continuous in each of the variables τ, x, ε relative to the remaining variables;

2) $|X(t, \tau, x, \varepsilon)| \leqslant M < \infty$, $(t, x) \in G$, $\varepsilon \in [0, \varepsilon_0]$;

3) uniformly for $t \in J$ the following limit exists:

$$\lim_{T\to\infty} \frac{1}{T} \int_{t}^{t+T} X(s, \tau, x, 0)\,du(s) = X(\tau, x), (\tau, x) \in G$$

4) the function $X(\tau, x)$ satisfies the Lipschitz condition in x for $x \in B_x(K)$, $\tau \in J$ with constant L, and uniformly continuous in τ relative to $x \in B_x(K)$;

5) the equation in R^n

$$dx/d\tau = X(\tau, x) \tag{3.4}$$

is a uniformly asymptotically stable solution $x = \psi(\tau, t_0, \xi_0)$ $(\psi(\varepsilon t_0, t_0, \xi_0) = \xi_0)$ (uniformly asymptotically stable in parts of the variables $x_1, \ldots, x_k$ $(k < n)$), which together with its certain ρ-neighbourhood $(\rho > 0)$ is contained in the set G.

Then for any α, $0 < \alpha < \rho$ numbers $\varepsilon_1(\alpha)$, $0 < \varepsilon_1 < \varepsilon_0$ and $\beta(\alpha)$ exist wuch that for all $0 < \varepsilon < \varepsilon_1$ the solution $\varphi(t, t_0, x_0)$ of (3.3), defined for $t \geqslant t_0$ and contained in $B_x(K)$, for which $|x_0 - \xi_0| < \beta(\alpha)$ satisfies the inequality

$$|\varphi(t, t_0, x_0) - \psi(\varepsilon t, t_0, \xi_0)| < \alpha \quad t \geqslant t_0$$
$$(|\varphi_i(t, t_0, x_0) - \psi_i(t, t_0, \xi_0)| < \alpha, \ i = 1, \ldots, k).$$

Proof. Making the change of time $\tau = \varepsilon t$ in (3.3), we obtain

$$Dx(\tau) = \varepsilon X(\tau/\varepsilon, \tau, x, \varepsilon)\, Du(\tau/\varepsilon) \tag{3.5}$$

where, for convenience, $x(\tau/\varepsilon)$ is again denoted by $x(\tau)$. We will show that the right side of (3.5) converges integrally to the function $X(\tau, x)$ defined in condition 3) of the theorem. Hence and from Theorem 1 the assertion of the theorem will result.

By virtue of the continuity of $X(\tau/\varepsilon, \tau, x, \varepsilon)$ in the fourth variable, that is uniform with respect to the remaining variables, it is sufficient to establish that for any $\delta > 0$ for sufficiently small ε

$$\Pi(\varepsilon) = \sup_{|t_2 - t_1| \leqslant 1} \left| \int_{t_1}^{t_2} \varepsilon X\left(\frac{\tau}{\varepsilon}, \tau, x, 0\right) du\left(\frac{\tau}{\varepsilon}\right) - \int_{t_1}^{t_2} X(\tau, x)\, d\tau \right| < \delta$$

We select a number $\eta > 0$ such that for $|\tau_1 - \tau_2| < \eta$ the following inequalities are satisfied

$$\left| X\left(\frac{\tau}{\varepsilon}, \tau_1, x, 0\right) - X\left(\frac{\tau}{\varepsilon}, \tau_2, x, 0\right) \right| < \frac{\delta}{4},$$
$$|X(\tau_1, x) - X(\tau_2, x)| < \frac{\delta}{4}$$

Let $g(\tau)$ be a peicewise-constant function defined in the interval $[t_1, t_2]$ $(|t_2 - t_1| \leqslant 1)$ such that $|\tau - g(\tau)| < \eta$, $\tau \in [t_1, t_2]$. Then

$$\Pi(\varepsilon) \leqslant \left| \int_{t_1}^{t_2} \varepsilon \left[X\left(\frac{\tau}{\varepsilon}, \tau, x, 0\right) - X\left(\frac{\tau}{\varepsilon}, g(\tau), x, 0\right) \right] du\left(\frac{\tau}{\varepsilon}\right) \right| +$$
$$\left| \int_{t_1}^{t_2} \varepsilon X\left(\frac{\tau}{\varepsilon}, g(\tau), x, 0\right) du\left(\frac{\tau}{\varepsilon}\right) - \int_{t_1}^{t_2} X(g(\tau), x)\, d\tau \right| +$$
$$\left| \int_{t_1}^{t_2} [X(g(\tau), x) - X(\tau, x)]\, d\tau \right| < \frac{\delta}{2} + \Lambda$$

$$\Lambda = \left| \sum_{k=1}^{n} \left[\int_{\sigma_{k-1}}^{\sigma_k} \varepsilon X\left(\frac{\tau}{\varepsilon}, \tau_k, x, 0\right) du\left(\frac{\tau}{\varepsilon}\right) - \int_{\sigma_{k-1}}^{\sigma_k} X(\tau_k, x)\, d\tau \right] \right|$$

where $t_1 = \sigma_0 < \sigma_1 < \ldots < \sigma_n = t_2$. It remains to show that $\Lambda < \delta/(2n)$ for sufficiently small ε. This follows from the limit equality $\lim \Lambda = 0$ as $\varepsilon \to 0$. The latter also follows from condition 3) of the theorem.

We note that in the case when the solution $\psi(\varepsilon t, t_0, \xi_0)$ of (3.4) is uniformly asymptotically stable in parts of the variables, it is necessary to use Remark 2° to Theorem 1.

4. The method described enables the question of the closeness between solutions of exact and averaged equations in an infinite interval in systems with fast and slow variables to be investigated. For instance, we examine the following system of differential equations with a rapid phase:

$$Dx(t) = \varepsilon X(x, y, \varepsilon)\, Du(y), \quad dy/dt = \omega(x) + \varepsilon Y(x, y, \varepsilon) \tag{4.1}$$

where x is a n-dimensional vector, y is a scalar variable, ε is a small positive parameter that changes in the interval $[0, \varepsilon_0]$, and $Dx(t), Du(y)$ are generalized derivatives of the functions $x(t)$ and $u(y)$. We assume that the function $X(x, y, \varepsilon)$ with values in R^n and the scalar function $Y(x, y, \varepsilon)$ are periodic in the variable y with period 2π, while $u(y)$ is a scalar 2π-periodic function of bounded variation. Later conditions are imposed on the functions $\omega(x)$ $Y(x, y, \varepsilon)$ such that $u(y(t))$ is a function of bounded variation.

Theorem 3. Suppose

1) the functions $X(x, y, \varepsilon)$, $Y(x, y, \varepsilon)$ are defined for $x \in B_x(K)$, $y \in (-\infty, \infty)$, $\varepsilon \in [0, \varepsilon_0]$, and continuous in the variables x, ε uniformly relative to the remaining variables;

2) the function $\omega(x)$ satisfies the inequality $\omega(x) \geqslant c > 0$ for $x \in B_x(K)$ where c is a certain constant, and the Lipschitz condition in x for $x \in B_x(K)$ with constant L;

3) a constant M exists such that

$$|X(x, y, \varepsilon)| \leqslant M, \; |Y(x, y, \varepsilon)| \leqslant M, \; x \in B_x(K), \; y \in (-\infty, \infty), \; \varepsilon \in [0, \varepsilon_0]$$

4) the function

$$X(x) = \frac{1}{2\pi}\int_0^{2\pi} X(x, y, 0)\, du(y)$$

satisfies the Lipschitz condition in x for $x \in B_x(K)$ with constant L_1;

5) the equation

$$dx/d\tau = X(x) \tag{4.2}$$

has a uniformly asymptotically stable solution $x = \psi(\tau, t_0, \xi_0)$ that belongs to the domain $B_x(K)$ together with its ρ-neighbourhood $(\rho > 0)$.

Then for any α, $0 < \alpha < \rho$ numbers $\varepsilon_1(\alpha)$ $0 < \varepsilon_1 < \varepsilon_0$ and $\beta(\alpha)$ exist such that for all $0 < \varepsilon < \varepsilon_1$ the slow variable solutions of system (4.1) for which $|x_0 - \xi_0| < \beta(\alpha)$, satisfy the inequality

$$|x(t, t_0, x_0, y_0) - \psi(\varepsilon t, t_0, \xi_0)| < \alpha, \; t \geqslant t_0$$

Proof. We make the changes $\tau = \varepsilon t, \alpha = \varepsilon y$ in system (4.1). We obtain the system

$$Dx(\tau) = \varepsilon X\left(x, \frac{\alpha}{\varepsilon}, \varepsilon\right) Du\left(\frac{\alpha}{\varepsilon}\right), \quad \frac{d\alpha}{d\tau} = \omega(x) + \varepsilon Y\left(x, \frac{\alpha}{\varepsilon}, \varepsilon\right) \tag{4.3}$$

It follows from conditions 2) and 3) of the theorem that for sufficiently small ε the function $\alpha(\tau)$ is monotonic, and therefore, for sufficiently small ε it is possible to take α as an independent variable instead of τ. System (4.3) is written in the new time as

$$\begin{aligned} Dx(\alpha) &= \varepsilon \frac{1}{\omega(x)} X\left(x, \frac{\alpha}{\varepsilon}, \varepsilon\right) Du\left(\frac{\alpha}{\varepsilon}\right) + \varepsilon^2 X_1\left(x, \frac{\alpha}{\varepsilon}, \varepsilon\right) Du\left(\frac{\alpha}{\varepsilon}\right) \\ \frac{d\tau}{d\alpha} &= \frac{1}{\omega(x)} + \varepsilon Y_1\left(x, \frac{\alpha}{\varepsilon}, \varepsilon\right) \end{aligned} \tag{4.4}$$

where the functions $X_1(x, \alpha/\varepsilon, \varepsilon)$, $Y_1(x, \alpha/\varepsilon, \varepsilon)$ possess the same properties as the corresponding functions without the subscript 1. It is seen that the right sides of system (4.4) converge integrally to the right sides of the system

$$\frac{dx}{d\alpha} = \frac{1}{\omega(x)} X(x), \quad \frac{d\tau}{d\alpha} = \frac{1}{\omega(x)} \tag{4.5}$$

which has the following form in time

$$dx/d\tau = X(x), \; d\alpha/d\tau = \omega(x) \tag{4.6}$$

The solution of system (4.6) corresponding to the solution $x = \psi(\tau, t_0, \xi_0)$ of (4.2) is evidently uniformly asymptotically stable in the variable x and the corresponding solution of system (4.5) possesses this same property. Applying Theorem 2 to system (4.4) we obtain the statement of the theorem.

Theorem 3 enables us to give a foundation to the applicability of the averaging method to quasiconservative vibration-impact systems since corresponding systems in /4/ results in the form (4.1).

REFERENCES

1. BIRD W.S., Stability under constantly acting perturbations and the averaging principle in an infinite interval. In: Thoery of Stability and Its Applications. Nauka, Novosibirsk, 1979.
2. BANFI C., Sull'appressimazione di processi non stazionari in mecanica non-lineare. Bull. Unione mate. Ital., 22, 4, 1967.
3. SETNA P.R., Systems with fast and slow time. Proc. Fifth Internat. Conf. on Non-linear Vibrations, 1, Izdat. Akad. Nauk UkrSSR, Kiev, 1970.
4. BABITSKII V.I., KOVALEVA A.S. and KRUPENIN V.L., Investigation of quasiconservative vibration-impact systems by the averaging method. Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, 1, 1982.
5. SAMOILENKO A.M., Averaging method in systems with jolts. Matematicheskaya Fizika, vyp, 9, Naukova Dumka, Kiev, 1971.
6. MITROPOL'SKII YU.A., The Averaging Method in Non-linear Mechanics. Naukova Dumka, Kiev, 1971.
7. PANDIT S.G. and DEO S.G., Differential systems involving impulses. Lecture Notes in Mathematics, 954, Springer Verlag, Berlin, 1982.
8. BARBASHIN E.A., Introduction to Stability Theory, Nauka, Moscow, 1967.

Translated by M.D.F.